Kant's Space and Modern Mathematics

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Mind, Vol. 2, No. 5. (Jan., 1877), pp. 38 – 46

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IV. KANT'S SPACE AND MODERN MATHEMATICS. (38)

The remarkable modern speculations concerning non-Euclidean sorts of space, of which Prof. Helmholtz gave some account in No. III. of MIND, are likely to be hailed as one of the chief difficulties with which the Kantian theory of space will have to deal. "If we can imagine such spaces of other sorts," that learned writer tells us, "it cannot be maintained that the axioms of geometry are necessary consequences of an *a priori* transcendental form of intuition, as Kant thought".

Before attempting to answer this argument, let me briefly point out a fundamental error that appears to hinder many adepts of positive science from realising the true nature of problems belonging to the theory of knowledge, or critical metaphysics.

In our wanderings on the border between science and philosophy we are apt to forget that it is impossible to move on both sides of the boundary line at once, and that whoever crosses it shifts his problem as well as his method. In physics (taking the word in its widest sense) we must adopt a standard of truth, which in philosophy is the very thing to be settled. When a sufficient amount of accurate observation has been digested by correct reasoning, we hold the result to be the adequate expression of real existence. We admit a real world, independent of all appearance to anybody's sense or reason, and take for its exact counterpart the world that offers itself to the *mens sana in corpore sano* after exhausting all the means of research at the command of mankind. Science has no suspicion of (39)
a distinction between 'objectivity' and 'reality'.

Of course the object of science is not altogether the same with that of popular belief. In every-day life we consider as real objects such things as appear to our senses, corrected by reasoning in the rough, as the blue firmament, the earth at rest, &c. In science the real object is what appears to be to the experienced mind attaining the very limit of its powers, and sensible phenomena sink into mere signs of the presence of certain objects. By interpretation of these signs the real object is attained. And if many a theory of the present day will probably be modified by ulterior investigation, still we are moving towards the end of representing the real object as it is.

Yet the real object of science has so much at least in common with that of ordinary life as is wanted for the purposes of measuring and calculation. It retains the space and time, the motion and, to a certain extent, even the matter and force of popular belief. It is not the object of pure thought, evolved from principles presupposed by necessity in every act of thinking, but of thought as applied to data of sense. However simplified by abstraction, it always bears the traces of its sensible origin.

In geometry proper, or constructive geometry (including stereometry), a great many qualities of things are disregarded, while it only attends to the space in which bodies appear to exist and move. But, however shorn, of qualities, its object is imagined as something to a certain extent analogous to what we see and touch. Hence its teachings may be assisted by diagrams and models, not mere conventional signs like those of arithmetic or logic. Because it takes from sense-intuition only the very first data, which are the same whatever part of our experience we proceed from, it assumes the aspect of a purely deductive science like arithmetic. Nevertheless its empirical basis may be shown by its inability to construct, for instance, an aggregate of four dimensions. Its real object is that of physics and of common life, considered exclusively as to the metrical proportions of figures imaginable in its space. To demand logical proof for genuine geometrical axioms is a mistake, because every proof must proceed from some ultimate premisses, which in this case must concern space. There are no data about space either in logic or arithmetic, but only in our sense-intuition, and precisely the data expressed in those axioms.

The algebraical geometry of modern science is algebra, a more general form of arithmetic, a series of speculations concerning quantities. Its sole connection with geometry is the understanding that the quantities it considers are meant as quantities of geometrical data; but this understanding is not embodied in the (40)
algebraical symbols themselves. As we learn from Prof. Helmholtz (l.c. p. 309), time as well as a line may be regarded here as an aggregate of one dimension, and the system of colours as an aggregate of three dimensions. The formulae and their analysis remain the same whether the aggregates be assumed to be spatial ones or of a different nature. Hence it is possible to pursue the chain of inference far beyond the limits of any geometrical interpretation, and even, by varying the premisses in which we express certain geometrical data, to prepare formulas that would apply to spaces foreign to our experience, provided any such could be conceived by the human imagination. The proof in this case is entirely logical: supposing certain relations of quantities, certain other relations must be admitted also, or there would be an end to all our thinking. However, the link between such a system of inferences and its application to qualities of either objective or assumed space is not comprehended in the system itself, but supplied from without, and it remains to be seen how much of the algebraical system, will bear

translation into geometry.

Now, when we aim at a theory of knowledge and enter into discussion with such thinkers as Berkeley, Hume or Kant, we find ourselves on a ground quite different from that of either physics or geometry. The notions of 'objectivity' and 'reality,' hitherto equivalent, must be carefully kept asunder, or else it becomes impossible even to understand the questions at issue. We must be prepared to examine opinions like these: that there is nothing real except mind, whereas space and bodies are merely its object; or that besides mind there is a reality, impressing it so as to produce an object wholly dissimilar from the reality itself. Again, if admitting impressions from without, we may have to enquire in how far the object is dependent on these and on the constitution of the mind respectively. If it were established beyond all doubt that the 'object' and the 'real' are one and the same, all examination of such questions and theories would become an empty ceremony, and the paradoxes of Idealism absurdities unworthy of our notice. But as things are now, results of scientific research involving that as- sumption cannot be rightly employed as evidence against philosophical tenets that disclaim its validity.

For a scientific man fresh from physiology of the senses, it is hard to keep in mind that the perceiving, imagining and thinking 'subject' of philosophy is not altogether the same as that with which he had to deal in his former pursuits. There he considered it as a unity of body and mind, one of a class of objects in the world we observe. Here, it is nothing more than the correlative of every (41)
object whatever, the observer and thinker opposed to them all. Unaccustomed to this kind of abstraction, the student of nature speedily-rounds it off into the full *anthropos* of physiology, not being aware that he has crossed the fatal border; and much of the reasoning current in his own domain is no longer acceptable as lawful tender.

From geometry proper, there is an easy transit into metaphysics, by the road of analytical geometry, which science has but a conventional connection with the data of intuition, and merges into pure arithmetic. In order to determine the relations between construction and analysis, some will attempt to reduce the latter to an abstraction from the sensible object like geometry, while others try to explain the foundations of geometry as necessities of thought unassisted by the senses. Both theories belong to the province of Philosophy; but from the familiar intercourse between mathematics and natural science, it is evident that Science has a great chance of being called in as arbiter and usurping the office without suspicion.

In the present case, the first question is whether any sort of space besides the space of Euclid be capable of being imagined. More than three dimensions, it is allowed, we are quite unable to represent. But we are told of spherical and pseudospherical space, and non-Euclideans exert all their powers to legitimate these as space by making them imaginable. We do not find that they succeed in

this, unless the notion of imaginability be stretched far beyond what Kantians and others understand by the word. To be sure, it is easy to imagine a spherical surface as a construction in Euclid's space; but we vainly attempt to get an intuition of a solid standing in the same relation to that surface as our own solids stand to the plane. A pseudospherical surface we may imagine; but then it is bounded by one or two edges. Nor is it of any avail to draw (as we are told) a piece from the edge back to the middle, and then continue it. This very operation betrays that the continuity of such a surface beyond the edge is not imaginable. We may cloak our perplexity by special phrases, saying that only limited strips of the surface can be "connectedly represented in our space," while it may yet be "thought of as infinitely continued in all directions". The former is just what is commonly understood by being 'imagined,' whereas being 'thought of' does not imply imagination any more than in the case of, say, $\sqrt{-1}$. And when we are assured that Beltrami has rendered relations in pseudospherical space of three dimensions imaginable by a process which substitutes straight lines for curves, planes for curved surfaces, and points on the surface of a finite sphere for infinitely distant points, we might as well believe that a cone is rendered (42)
sufficiently imaginable to a pupil by merely showing its projection upon a plane as a circle or a triangle. Just the characteristic features of the thing we are to imagine must be done away with, and all we are able to grasp with our intuition is a translation of that thing into something else. As to the image in a convex mirror, referred to by Prof. Helmholtz in his article, we do not mentally contrast it with our objective world in Euclidean space, but only with the habitual aspect of that world as seen from a given point of view. In the latter also things appear to contract as they retire to a distance. Only we have learned to conceive the objective space as one in which we ourselves are able to move in all directions and shift our point of view at pleasure. So with some practice we actually see those things not growing smaller, but moving away from the place where we may happen to be. The world in the mirror offers itself as a novel aspect of the same world, needing a larger amount of practice for its interpretation, because complicated by unwonted circumstances. As a form of the objective world, which remains the same from whatever point we inspect it, we can imagine, not any space in which motion implies flattening or change of form of any kind, but only the space known from our sense-experience, the space of Euclid. All other 'space' contrived by human ingenuity may be an aggregate with fictitious properties and a consistent algebraical analysis of its own, but space it is called only by courtesy.

Even admitting for a moment that our mind is capable of *imagining* different sorts of space, it might still be maintained that the only possible form of *actual intuition* for a mind like ours, as affected by real things outside of it, is Euclidean space. When we hold the origin of our geometrical axioms to be empirical, it does not follow that a real space must be assumed as being transported in some way through organs of sense into the percipient mind. Of experience itself there are different explanations, as far as explanations go. Granted that

I take my 'flat space' from my perceptions, and these are forced upon me by something not myself, variety of perceptions ought to originate in a variety of outward impulses. But then perception may be, for aught I know, wholly dissimilar in nature from both the impulse and that which produced the impulse, as the perception of red or blue is believed to be the effect of certain undulations in the optic nerve, produced in their turn by the waves termed light, and yet not to be compared with either. Our intuition of space may be empirical without a real space to correspond, provided there be any reality whatever compelling the mind to exert its native powers in constructing space as we know it, which the (43)
mind would not do unless so compelled. In that case space, Euclid's space, would remain a form of intuition, *a priori* and transcendental.

We read that "geometrical axioms must vary according to the kind of space *inhabited*". Why this must be, one cannot understand, unless it be proved first, which is not proved at all, that space as represented by a sentient being is necessarily a copy of a space in which it lives and moves. Even if we suppose that the subject resides in a real space, and that its intuitions of space depend entirely on what it *perceives*, the question remains, how much of its perception is due to the constitution of the subject itself, and how much to impressions from the outer world? Also, what is the relation between those impressions and the spatial arrangement of that world? The space represented on the faith of perception might yet be different from the real space. Nay, on the popular empirical ground taken by physiology, the proposition is a disputable one. Dr. Mises (Prof. Fechner), in one of his witty paradoxes of thirty years ago, reprinted last year in his *Kleine Schriften*, supposed reasoning beings of two dimensions only, like the men we see in the *camera obscura*, who move together with the plane which they inhabit through a third dimension, and perceive that movement only as a continued series of changes in their superficial universe. By analogy he started the hypothesis of a fourth dimension through which we might be moving our- selves. Now we know that analytical geometry is ready to grapple with any number of dimensions,[1] though they can never be imagined. These plane-people of Mises are quite as imaginable as the sphere-dwellers of Prof. Helmholtz. They would really exist in a space of three dimensions, inhabiting two of them and moving through the third, yet perceiving but two of them as dimensions. So would the sphere-dwellers; for the surface of a sphere means either nothing at all, or the boundary of a solid of three dimensions. Only in their case the third dimension would influence their intuition by preventing them, for instance, from ever gathering experience of parallel lines and geometrical similarity between figures of different size.[2] However, as our mathematicians succeed in explaining

[1]Cf. the *Ausdehnungslehre* of Hermann Grassmann.

[2]Unless, indeed, they were small enough to perceive only a very limited portion of their surface, which might easily impress them as flat, as our earth did the first Greek philosophers. We need not stop to inquire whether we ourselves ever get sense-experience of undoubtedly parallel lines. Nevertheless such are constructible out of primary elements supplied by sense-

properties of spaces unknown to. our experience, even of those of four and more dimensions, there is no reason to deny the same faculty to our imaginary surface-men. As all straightest lines on a sphere end by meeting somewhere, why should they not for once suppose a different surface, on which straightest lines might be drawn in any direction so as to retain the same distance to infinity, and, reasoning on this and a few more suppositions, discover the analytical geometry of the plane? Combining this with their original spherical theorems, some genius among them might conceive the bold hypothesis of a third dimension, and demonstrate that actual observations are perfectly explained by it. Henceforth there would be a double set of geometrical axioms; one the same as ours, belonging to science, and another resulting from experience in a spherical surface only, belonging to daily life. The latter would express the 'object' of sense-intuition; the former, 'reality,' incapable of being represented in empirical space, but perfectly capable of being thought of and admitted by the learned as real, albeit different from the space inhabited. (44)

The 'rigidity' ascribed to geometrical figures is hardly to be considered as a physical quality. A physical solid, say an india-rubber ball, may be thought of as being flattened to a spheroid or a disc, and still retain its identity, because the matter remains the same. It would be perfectly rigid in a physical sense, if its form were unchangeable by any external force whatever. But a geometrical sphere is the same only as long as both its form and size remain what they are. The rigidity is not resistance against force, but simple identity with itself. We might conceive a spheroid of the same volume, and an unbroken series of spheroids between it and the sphere; so by analogy with the physical body we might say that the sphere was gradually flattened to the ultimate form in the series. Still in the geometrical sense there would be no identity between the sphere and any of the spheroids, because here matter is wanting, but only a successive substitution of something else instead of the primitive figure. If we apply one sphere to another, and find out their congruence or the reverse, the meaning is not that a physically rigid body is to be transported through all the ntervening parts of space. The purpose is answered as well by mentally cancelling the old sphere, and constructing a new one on the same principle and with the same radius, so that its centre coincide with that of the sphere to be compared with it. In the case of mechanical science deciding that- two bodies must have varied in the same sense during such an operation, the inference would be that the consequences of geometrical application of figures to each other can never be verified by actual experience on physical bodies for that reason, to say nothing (45)
of their impenetrability. But geometry would declare bodies liable to vary, to be different from its own solids. Of course, its own abstract notions of space and figure may be supplemented at pleasure by taking into account time of movement, or a concept of matter just sufficient to distinguish a filled part of space from

intuition.

an empty one. In the former case we come to phoronomy, in the latter to mixed geometrical speculations about bodies capable of contractions and distensions. Such speculations are as lawful as what most people understand by geometry, and it appears that physicists find them useful for their ulterior purposes. Only they must not be confounded with the doctrine of space and its measures, in which a solid is simply a part of space of a certain form and size, a surface the boundary between such parts and so on. These parts of space it would be absurd to consider as changeable, whatever experience may affirm concerning physical bodies that move in space. It is certainly true in one sense, that the axioms of geometry "merely define what qualities and deportment a body must have to be recognised as rigid". But this regards geometry as applicable to bodies or material things; its own solids are not meant either to have or to lack physical rigidity.

Nevertheless geometrical axioms are synthetic propositions, because they are not to be deduced by pure logic from the definition of their subject-terms, but are found by intuition of the space offered to us as a form of our objective world. As far as we know, that world and its space could be quite different from what they are, were it not for sense-experience which supplies the first elements of construction, and reflection which constructs figures and examines them as if actually seen. The axioms of geometry proper are discoveries resulting from the contemplation of objective space by itself; as soon as we add the empirical elements of movement, properly so called, of bodies filling space, &c., we stand upon another ground.

To conclude these observations, the Kantian, theory of space, as defined by Prof. Helmholtz himself, contains three distinct assertions: —

(a.) Space is a form of ***intuition***: any conception of ours must; be imaginable to be what we call space. [This is admitted by the opponents; only, non-Euclideans try to make imaginable that which is not so in the sense required for argumentation in this case.]

(b.) Space is a form ***a priori***: a native form of our perceptive faculty, not a datum passively received from without. [The opponents attempt to refute this by proving the empirical origin of our notions of space.. Between this proof and the refutation of Kant's assertion there is wanting the proof that empirical knowl- (46)
edge is acquired by simple importation or by counterfeit, and not by peculiar operations of the mind solicited by varied impulses from an unknown reality.]

(c.) Space is a ***transcendental*** form: belonging to our own object by some necessity arising from the unknown constitution of our mind; but not therefore belonging to the real world as well. [The opponents overlook the distinction between 'objectivity' and 'reality', and reason, as they would do in physical science, on the tacit supposition of the two being identical, and Kant's assertion disproved beforehand.]

After this, the final propositions of the article in question would have to be modified as follows : —

(1.) The axioms of geometry, taken by themselves out of all connection with mechanical propositions, represent no relations of physical objects. When strictly isolated, if we regard them with Kant as forms of intuition transcendentally given, they contribute a form into which any empirical content whatever will fit, and which therefore does not in any way limit or determine beforehand the nature of that content. In other words, axioms concerning parts of space do not determine the deportment of bodies filling such parts at a given moment. We may admit that this would hold true if the axioms given were those of spherical or pseudospherical geometry; however, the (possibly transcendental) . form of intuition actually given.is that analysed in Euclid's axioms.

(2.) As soon as certain principles of mechanics are conjoined with the axioms of geometry, we obtain a system of propositions which has full objective or physical import, and which can be verified or overturned by fresh sense-observations, as from sense-experience it can be inferred. If such a system were to be taken as a transcendental form of intuition and thought, there must be assumed a constancy of laws determining the relations between the mind's objects and the impulses which it receives from an unknown reality.

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Leyden, *Sept. 30,1876.*